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ON VORTEX MOTION IN A VISCOUS FLUID

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The generalized Green formulas which I derived in my paper entitled "Sur les formules de Green généralisées qui se présentent dans l'hydrodynamique et sur quelques-unes de leurs applications" (Act a Math. vol. 34) are the basis of a theory of vortex motion in an incompressible viscous fluid. What I shall give here is only an application of these formulas for the very simplest cases. One of the formulas interests me here because of its great simplicity. It is only slightly more complicated than the corresponding Helmholtz formula, and is valid in this connection; i.e. it is an exact conclusion of the Navier differential equations.

1. Here I intend to deal with the following problem. At a given moment, $t = 0$, in an infinitely extensive fluid, let w equal zero, and $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \bar{w}$ be a continuous, continuously differentiable function of R in a cylinder $R = \sqrt{x^2 + y^2} = R_0$ which vanishes for $R = R_0$ and only for $R = R_0$. (Outside the cylinder let $R = 0$.) When $t = 0$ the motion of the fluid is a Helmholtz vortex motion about a cylindrical vortex. The question is, how does the motion develop from this initial state if no forces are acting upon the fluid.

To answer this question let us consider the differential equations for the two-dimensional case:

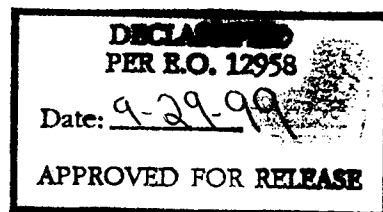
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \mu \Delta u,$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = - \frac{\partial p}{\partial y} + \mu \Delta v,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \text{ and, } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

These equations can be put in a simpler form:

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$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= v\bar{w} - \frac{\partial g}{\partial x} + \mu \Delta u, \\ \frac{\partial v}{\partial t} &= -u\bar{w} - \frac{\partial g}{\partial y} + \mu \Delta v, \\ \bar{w} &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \text{ and } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \end{aligned} \right\} \quad (1)$$

Here:

$$g = p + \frac{1}{2}(u^2 + v^2).$$

In the previously mentioned paper I proved the following principle: if system 1, (where $v\bar{w}$ and $-u\bar{w}$ are considered known functions) has solutions which satisfy inequalities of the form

$$\left. \begin{aligned} |u|, |v| &< \frac{M}{(1+R)^\alpha}, \quad \left| \frac{\partial u}{\partial x} \right|, \left| \frac{\partial v}{\partial y} \right| < \frac{M}{(1+R)^{1+\alpha}} \\ |p| &< m(1+R)^{1-\alpha}, \quad \alpha > 0. \end{aligned} \right\} \quad (2)$$

then these solutions must satisfy the equations:

$$\begin{aligned} u(x, y, t) &= \frac{1}{2\pi} \iint \left[u(\xi, \eta, \tau) u'_\xi + v(\xi, \eta, \tau) v'_\xi \right]_{\tau=0} d\xi d\eta \\ &+ \frac{1}{2\pi} \int_0^t d\tau \iint \left[v(\xi, \eta, \tau) u'_\xi - u(\xi, \eta, \tau) v'_\xi \right] \bar{w}(\xi, \eta, \tau) d\xi d\eta \end{aligned}$$

$$\begin{aligned} v(x, y, t) &= \frac{1}{2\pi} \iint \left[u(\xi, \eta, \tau) u'_\eta + v(\xi, \eta, \tau) v'_\eta \right]_{\tau=0} d\xi d\eta \\ &+ \frac{1}{2\pi} \int_0^t d\tau \iint \left[v(\xi, \eta, \tau) u'_\eta - u(\xi, \eta, \tau) v'_\eta \right] \bar{w}(\xi, \eta, \tau) d\xi d\eta \end{aligned}$$

Here we have

$$u'_\xi = \frac{\partial^2 p}{\partial \eta^2}, \quad v'_\xi = \frac{\partial^2 p}{\partial \xi \partial \eta} = u'_\eta, \quad v'_\eta = \frac{\partial^2 p}{\partial \xi^2},$$

$$p = - \int_0^{\infty} e^{-\frac{r^2}{4\mu(t-\tau)}} \frac{d\tau}{\tau} - \log r, \quad r^2 = (x-\xi)^2 + (y-\eta)^2.$$

The integrals on the right can be substantially simplified. Integration by parts gives us:

$$\begin{aligned} \iint \left[u(\xi, \eta, \tau) u'_\xi + v(\xi, \eta, \tau) v'_\xi \right]_{\tau=0} d\xi d\eta &= - \iint \left(\frac{\partial v}{\partial \xi} - \frac{\partial u}{\partial \eta} \right)_{\tau=0} \left(\frac{\partial p}{\partial \eta} \right)_{\tau=0} d\xi d\eta \\ &= - \iint \bar{w}(\xi, \eta, 0) \left(\frac{\partial p}{\partial \eta} \right)_{\tau=0} d\xi d\eta \end{aligned}$$

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$$\iint_{R \in R_0} \{ u(\xi, \eta, \tau) u'_\eta + v(\xi, \eta, \tau) v'_\xi \} \bar{w} d\xi d\eta = \iint_{R \in R_0} \bar{w}(\xi, \eta, 0) \left(\frac{\partial P}{\partial \xi} \right)_{\tau=0} d\xi d\eta.$$

If we extend integration to a finite region with the limiting curve S, we have:

$$\iint \{ v(\xi, \eta, \tau) u'_\xi - u(\xi, \eta, \tau) v'_\xi \} \bar{w} d\xi d\eta = \iint \frac{\partial P}{\partial \eta} \left[\bar{w} \left(\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} \right) + u \frac{\partial \bar{w}}{\partial \xi} + v \frac{\partial \bar{w}}{\partial \eta} \right] d\xi d\eta + \int_S \frac{\partial P}{\partial \eta} \bar{w} (u \cos \eta x + v \cos \eta y) dS.$$

Now:

$$\frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} = 0.$$

For reasons of symmetry it is evident that the particles of fluid describe circles about the Z-axis and that even for $t > 0$, $\bar{w}$ can depend only on R.

Therefore:

$$\frac{u}{\eta} = - \frac{v}{\xi}, \quad \frac{\frac{\partial \bar{w}}{\partial \xi}}{\xi} = \frac{\frac{\partial \bar{w}}{\partial \eta}}{\eta}$$

Consequently:

$$u \frac{\partial \bar{w}}{\partial \xi} + v \frac{\partial \bar{w}}{\partial \eta} = 0.$$

Finally we show that the last integral on the right converges toward zero if the curve S moves toward infinity, on account of the inequalities 2. Then we have:

$$\iint \{ v(\xi, \eta, \tau) u'_\xi - u(\xi, \eta, \tau) v'_\xi \} \bar{w} d\xi d\eta = 0,$$

and likewise:

$$\iint \{ v(\xi, \eta, \tau) u'_\eta - u(\xi, \eta, \tau) v'_\eta \} \bar{w} d\xi d\eta = 0.$$

Thus we have obtained the formulas:

$$u(\xi, \eta, t) = - \frac{1}{2\pi} \iint_{R \in R_0} \bar{w}_{\tau=0} \left(\frac{\partial P}{\partial \eta} \right)_{\tau=0} d\xi d\eta =$$

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$$= \frac{1}{2\pi} \iint_{R \leq R_0} \left(1 - e^{-\frac{r^2}{4\mu t}}\right) \bar{w}(\xi, \eta, 0) \frac{r^2}{r^2} d\xi d\eta$$

$$v(x, y, t) = \frac{1}{2\pi} \iint_{R \leq R_0} \bar{w}_{t=0} \left(\frac{\partial P}{\partial \xi}\right) d\xi d\eta \quad (3)$$

$$= -\frac{1}{2\pi} \iint_{R \leq R_0} \left(1 - e^{-\frac{r^2}{4\mu t}}\right) \bar{w}(\xi, \eta, 0) \frac{\xi - x}{r^2} d\xi d\eta.$$

These are the formulas I emphasized in the introduction on account of their striking simplicity. They differ from those in the Helmholtz theory only by the factor $1 - e^{-\frac{r^2}{4\mu t}}$.

If we consider a point in the fluid whose distance from the initial vortex is large, we can put $r = \text{const.}$ $\frac{\xi}{r} = \frac{\eta}{r} = 0$ in the integrals.

We get:

$$\left. \begin{aligned} u(x, y, t) &= \frac{I_0}{2\pi} \frac{y}{R^2} \left(1 - e^{-\frac{R^2}{4\mu t}}\right), \\ v(x, y, t) &= \frac{I_0}{2\pi} \frac{x}{R^2} \left(1 - e^{-\frac{R^2}{4\mu t}}\right), \end{aligned} \right\} \quad (4)$$

$$R^2 = x^2 + y^2, \quad I_0 = \iint_{R \leq R_0} \bar{w}(\xi, \eta, 0) d\xi d\eta.$$

I_0 is the measure of the intensity of the vortex at $t = 0$.

From this formula we see that the velocity at which the particles describe their circular paths about the initial vortex, at a distance from the vortex which is large compared with the section of the vortex, is equal to

$$\frac{I_0}{2\pi R} \left(1 - e^{-\frac{R^2}{4\mu t}}\right). \quad (5)$$

The time which must elapse before this velocity has fallen to half its initial value is determined from the equation:

$$1 - e^{-\frac{R^2}{4\mu t}} = \frac{1}{2}$$

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We get:

$$t = \frac{R^2}{2,772\mu}$$

If we try to determine the points where the velocity according to formula (5) is largest, we get the equation:

$$1 + \frac{R^2}{4\mu t} = e^{\frac{R^2}{4\mu t}}$$

The distance of these points from the center of the vortex is therefore proportional to $\sqrt{\mu t}$.

The intensity of the vortex at distance R is determined from the equation:

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{I_0}{4\pi\mu t} e^{-\frac{R^2}{4\mu t}}$$

Therefore, it has its maximum in the initial vortex.

The qualitative content of formulas (6) can be understood in the following manner: from the center of the initial vortex a motion is propagated in the fluid.

The curl of this motion has the same sign as the vortex everywhere outside the initial vortex, but its velocity is opposite to that of the original vortex.

2. Now I will assume that we have two straight parallel vortex filaments in a fluid. There will be no mention of an exact solution of the problem at this time. Our object is to come closer to reality than the Helmholtz theory permits. Therefore I will assume the latter to be correct in the first approximation. Expressing it more exactly, I assume that in a first approximation the motion everywhere outside the two vortex filaments is irrotational. We can also assume that the motion in the vortex filaments is independent of time and satisfies the conditions set up above. I assume that the axes of the vortex move in the manner prescribed by the Helmholtz theory, i.e. in circles about a common center, or if the intensities of the two vortices are equal with opposite direction the axes move along two parallel lines.

To carry the approximation one step further, we will introduce for $\vec{v}$ and $\vec{u}$ in formulas (1) the values given by the Helmholtz theory. Then we proceed in the same way as before, i.e. we consider the

$$u(x, y, t) = \frac{1}{2\pi} \iint [u(\xi, \eta, \tau) u'_\xi + v(\xi, \eta, \tau) v'_\xi]_{\tau=0} d\xi d\eta \\ + \frac{1}{2\pi} \int_0^t d\tau \iint [v(\xi, \eta, \tau) u'_\xi - u(\xi, \eta, \tau) v'_\xi] \bar{w}(\xi, \eta, \tau) d\xi d\eta$$

$$v(x, y, t) = \frac{1}{2\pi} \iint [u(\xi, \eta, \tau) u'_\eta + v(\xi, \eta, \tau) v'_\eta]_{\tau=0} d\xi d\eta \\ + \frac{1}{2\pi} \int_0^t d\tau \iint [v(\xi, \eta, \tau) u'_\eta - u(\xi, \eta, \tau) v'_\eta] \bar{w}(\xi, \eta, \tau) d\xi d\eta$$

These functions, $u(x, y, t)$ and $v(x, y, t)$, are the expressions valid in the second approximation for the components of the velocity. To put them in simpler form, let us consider the integral:

$$\int_0^t d\tau \iint [v(\xi, \eta, \tau) u'_\xi - u(\xi, \eta, \tau) v'_\xi] \bar{w}(\xi, \eta, \tau) d\xi d\eta,$$

the integration over the first vortex filament having been carried through.

According to my hypothesis, $\bar{w}(\xi, \eta, \tau)$ depends on τ only in that the position of the vortex is dependent on τ . In other words, we have:

$$\bar{w}(\xi, \eta, \tau) = \bar{w}(\xi - \xi_1, \eta - \eta_1, 0)$$

where ξ_1, η_1 are the coordinates of the axis of the first vortex filament.

Furthermore, we have:

$$u(\xi, \eta, \tau) = u(\xi_1, \eta_1, \tau) + \left(\frac{\partial u}{\partial \xi}\right)_1 (\xi - \xi_1) + \left(\frac{\partial u}{\partial \eta}\right)_1 (\eta - \eta_1) + \dots \\ v(\xi, \eta, \tau) = v(\xi_1, \eta_1, \tau) + \left(\frac{\partial v}{\partial \xi}\right)_1 (\xi - \xi_1) + \left(\frac{\partial v}{\partial \eta}\right)_1 (\eta - \eta_1) + \dots$$

Here we have:

$$u(\xi_1, \eta_1, \tau) = \frac{\partial \xi_1}{\partial \tau}, \quad u(\xi, \eta, \tau) = \frac{\partial \eta_1}{\partial \tau}$$

From our integral we obtain a term which can be described as follows:

$$- \int_0^t d\tau \iint \left(\frac{\partial^2 P}{\partial \xi \partial \eta} \cdot \frac{d\xi_1}{d\tau} + \frac{\partial^2 P}{\partial \eta^2} \cdot \frac{d\eta_1}{d\tau} \right) \bar{w}_1(\xi, \eta, \tau) d\xi d\eta$$

Now let us take the expression:

$$\frac{\partial}{\partial \tau} \iint \frac{\partial P}{\partial \eta} \bar{w}_1(\xi, \eta, \tau) d\xi d\eta$$

For $\tau \neq \tau$ we have:

$$\begin{aligned} \frac{\partial}{\partial \tau} &= \iint \frac{\partial^2 P}{\partial \eta^2} \bar{w}_1(\xi, \eta, \tau) d\xi d\eta = \iint \left(\frac{\partial^2 P}{\partial \eta^2 \partial \tau} \bar{w}_1 + \frac{\partial P}{\partial \eta} \cdot \frac{\partial \bar{w}_1}{\partial \tau} \right) d\xi d\eta = \\ &= \iint \left[\frac{\partial^2 P}{\partial \eta^2 \partial \tau} \bar{w}_1 - \frac{\partial P}{\partial \eta} \left(\frac{\partial \bar{w}_1}{\partial \xi} \cdot \frac{\partial \xi}{\partial \tau} + \frac{\partial \bar{w}_1}{\partial \eta} \cdot \frac{\partial \eta}{\partial \tau} \right) \right] d\xi d\eta = \\ &= \iint \left[\frac{\partial^2 P}{\partial \eta^2 \partial \tau} \bar{w}_1 + \left(\frac{\partial^2 P}{\partial \xi \partial \eta} \cdot \frac{\partial \xi}{\partial \tau} + \frac{\partial^2 P}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial \tau} \right) \bar{w}_1 \right] d\xi d\eta. \end{aligned}$$

The term of our integral obtained above therefore has the value:

$$\begin{aligned} \int_0^\tau d\tau \iint \frac{\partial^2 P}{\partial \eta^2 \partial \tau} \bar{w}_1 d\xi d\eta - \int_0^\tau d\tau \frac{\partial}{\partial \tau} \iint \frac{\partial P}{\partial \eta} \bar{w}_1 d\xi d\eta = \\ = \int_0^\tau d\tau \iint \frac{\partial^2 P}{\partial \eta^2 \partial \tau} \bar{w}_1 d\xi d\eta + \iint \left(\frac{\eta - \eta}{r^2} \bar{w}_1 \right)_{\tau=\tau} d\xi d\eta + \iint \left(\frac{\partial P}{\partial \eta} \bar{w}_1 \right)_{\tau=0} d\xi d\eta. \end{aligned}$$

We obtain an integral of similar structure, which is based on the second vortex filament. In this way we exhaust everything we obtain from the integral:

$$\int_0^\tau d\tau \iint \left[v(\xi, \eta, \tau) u'_\xi - u(\xi, \eta, \tau) v'_\xi \right] \bar{w}(\xi, \eta, \tau) d\xi d\eta.$$

On account of the symmetry which we assumed in the first approximation about the axes of the vortex filaments, the remaining parts of the integral vanish, as did the whole integral in the first case we discussed.

We transform the two first integrals in our formulas in the same way as in the preceding section.

We may group our results in the following formulas:

$$\begin{aligned} u(x, y, t) &= \frac{1}{2\pi} \iint \left[\frac{\eta - \eta}{r^2} \bar{w}_1 \right]_{\tau} d\xi d\eta + \frac{1}{2\pi} \iint \left[\frac{\eta - \eta}{r^2} \bar{w}_2 \right]_{\tau} d\xi d\eta \\ &+ \frac{1}{2\pi} \int_0^\tau d\tau \iint \frac{\partial^2 P}{\partial \eta^2 \partial \tau} \bar{w}_1(\xi, \eta, \tau) d\xi d\eta \\ &+ \frac{1}{2\pi} \int_0^\tau d\tau \iint \frac{\partial^2 P}{\partial \eta^2 \partial \tau} \bar{w}_2(\xi, \eta, \tau) d\xi d\eta. \end{aligned}$$

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$$V(x, y, t) = -\frac{1}{2\pi} \int \int_1 \left[\frac{\xi-x}{r^2} \bar{w}_1 \right]_t d\xi d\eta - \frac{1}{2\pi} \int \int_2 \left[\frac{\xi-x}{r^2} \bar{w}_2 \right]_t d\xi d\eta \\ - \frac{1}{2\pi} \int_0^t d\tau \int \int_1 \frac{\partial^2 P}{\partial \xi \partial \tau} \bar{w}_1(\xi, \eta, \tau) d\xi d\eta \\ - \frac{1}{2\pi} \int_0^t d\tau \int \int_2 \frac{\partial^2 P}{\partial \xi \partial \tau} \bar{w}_2(\xi, \eta, \tau) d\xi d\eta.$$

When the value of P is inserted we get:

$$u(x, y, t) = \frac{1}{2\pi} \int \int_1 \left[\frac{\eta-y}{r^2} \bar{w}_1 \right]_t d\xi d\eta + \frac{1}{2\pi} \int \int_2 \left[\frac{\eta-y}{r^2} \bar{w}_2 \right]_t d\xi d\eta - \frac{1}{8\pi\mu} \int_0^t \\ \int \int_1 \frac{\eta-y}{(t-\tau)^2} e^{-\frac{r^2}{4\mu(t-\tau)}} \bar{w}_1(\xi, \eta, \tau) d\xi d\eta \\ - \frac{1}{8\pi\mu} \int_0^t d\tau \int \int_2 \frac{\eta-y}{(t-\tau)^2} e^{-\frac{r^2}{4\mu(t-\tau)}} \bar{w}_2(\xi, \eta, \tau) d\xi d\eta.$$

$$v(x, y, t) = -\frac{1}{2\pi} \int \int_1 \left[\frac{\xi-x}{r^2} \bar{w}_1 \right]_t d\xi d\eta - \frac{1}{2\pi} \int \int_2 \left[\frac{\xi-x}{r^2} \bar{w}_2 \right]_t d\xi d\eta \\ + \frac{1}{8\pi\mu} \int_0^t d\tau \int \int_1 \frac{\xi-x}{(t-\tau)^2} e^{-\frac{r^2}{4\mu(t-\tau)}} \bar{w}_1(\xi, \eta, \tau) d\xi d\eta \\ + \frac{1}{8\pi\mu} \int_0^t d\tau \int \int_2 \frac{\xi-x}{(t-\tau)^2} e^{-\frac{r^2}{4\mu(t-\tau)}} \bar{w}_2(\xi, \eta, \tau) d\xi d\eta$$

We see at once that if the terms relative to the second vortex filament are omitted in these formulas and the first vortex filament is assumed to be fixed in space, the formulas obtained change into the valid formulas derived in the preceding section. This is a test of the correctness of our calculations.

Due to the complicated structure of our formulas we will limit ourselves to a qualitative discussion. If the fluid were free of friction, the maximum position of intensity of the vortex, as well as the fluid in the two vortices, would move in accordance with the law we postulated in the first approximation for the axes of the vortex. The result of the friction is that the maximum positions of the

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intensity shift, and the fluid in the vortices move relative to them. First we will investigate the latter phenomenon. This involves a study of how the movement in the middle of the part of the fluid which at a given instant is where a vortex should be (according to the Helmholtz theory) differs from the movement which Helmholtz maintains, every vortex ought to have. We see that the two first terms on the right in our formula for u and v correspond with those given by Helmholtz. The deviation therefore comes from the last two terms. Now if we consider the first vortex filament it is evident that the integrals coming from it must generally outweigh the integrals from the second vortex filament. The contents of these integrals can be expressed as follows: from each point of its path which goes beyond the vortex a movement is propagated whose curl outside the immediate vicinity of the point has the same sign as the vortex itself. The velocity of this movement, however, is opposite to the velocity of the vortex which would prevail at the moment when the vortex is at the point in question, according to Helmholtz. The strength of the movement diminishes very fast when we leave its center. Besides, it diminishes fast as t increases. From this it follows that the part of the fluid in the Helmholtz vortex at a given time must withdraw from the vortex in the direction in which the fluid behind the Helmholtz vortex moves. If we examine the integral coming from the second vortex filament in the same way, we find that the fluid behind the Helmholtz vortex must remain behind. The first part of our result can be stated in this way. When the vortices have the same sign the particles of the vortex move as though a repulsion were occurring between them. They move as if an attraction were taking place, when the vortices have opposite signs.

The position of the points of greatest vortex intensity is a more difficult problem. It hardly seems possible to draw general conclusions from the formulas. Outside the Helmholtz vortex we have:

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$$\begin{aligned} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= \frac{1}{4\pi\mu} \int_0^t d\tau \iint_1 \left(\frac{r^2}{4\mu(t-\tau)} - 1 \right) \frac{e^{-\frac{r^2}{4\mu(t-\tau)}}}{(t-\tau)^2} \bar{w}_1(\xi, \eta, \tau) d\xi d\eta \\ &\quad + \frac{1}{4\pi\mu} \int_0^t d\tau \iint_2 \left(\frac{r^2}{4\mu(t-\tau)} - 1 \right) \frac{e^{-\frac{r^2}{4\mu(t-\tau)}}}{(t-\tau)^2} \bar{w}_2(\xi, \eta, \tau) d\xi d\eta \\ &= -\frac{1}{4\pi\mu} \int_0^t d\tau \iint_1 \frac{\partial}{\partial \tau} \frac{e^{-\frac{r^2}{4\mu(t-\tau)}}}{t-\tau} \bar{w}_1(\xi, \eta, \tau) d\xi d\eta \\ &\quad - \frac{1}{4\pi\mu} \int_0^t d\tau \iint_2 \frac{\partial}{\partial \tau} \frac{e^{-\frac{r^2}{4\mu(t-\tau)}}}{t-\tau} \bar{w}_2(\xi, \eta, \tau) d\xi d\eta. \end{aligned}$$

In the First Vortex Filament we have:

$$\begin{aligned} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} &= \bar{w}_1(x, y, t) - \frac{1}{4\pi\mu} \int_0^t d\tau \iint_1 \frac{\partial}{\partial \tau} \frac{e^{-\frac{r^2}{4\mu(t-\tau)}}}{t-\tau} \bar{w}_1(\xi, \eta, \tau) d\xi d\eta \\ &\quad - \frac{1}{4\pi\mu} \int_0^t d\tau \iint_2 \frac{\partial}{\partial \tau} \frac{e^{-\frac{r^2}{4\mu(t-\tau)}}}{t-\tau} \bar{w}_2(\xi, \eta, \tau) d\xi d\eta. \end{aligned}$$

The factor $\frac{r^2}{4\mu(t-\tau)} - 1$, which is decisive for the sign of the integrand in our first formula for $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$, is positive for large values of r or small values of $(t-\tau)$. For each value of r the factor becomes negative for sufficiently large values of $(t-\tau)$. Thus we are directed to the rather interesting question, whether $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ can have the opposite sign of $\bar{w}_1$ and $\bar{w}_2$ if the latter have the same sign. At the present time I am unable to answer this question.

If x, y is a point whose distance from the vortices in the time interval 0 to t is large compared to the initial sections of the vortices, we can write our formulas as follows:

$$u(x, y, t) = \frac{I_1}{2\pi} \frac{y - \eta_1 r}{r^2} - \frac{I_2}{2\pi} \frac{y - \eta_2 r}{r^2} + \frac{I_1}{8\pi\mu} \int_0^t \frac{y - \eta_1 \tau}{(\tau - r)^2} e^{-\frac{r^2}{4\mu(t-\tau)}} d\tau + \frac{I_2}{8\pi\mu} \int_0^t \frac{y - \eta_2 \tau}{(\tau - r)^2} e^{-\frac{r^2}{4\mu(t-\tau)}} d\tau$$

$$v(x, y, t) = \frac{I_1}{2\pi} \frac{x - \xi_1 r}{r^2} + \frac{I_2}{2\pi} \frac{x - \xi_2 r}{r^2} - \frac{I_1}{8\pi\mu} \int_0^t \frac{x - \xi_1 \tau}{(\tau - r)^2} e^{-\frac{r^2}{4\mu(t-\tau)}} d\tau - \frac{I_2}{8\pi\mu} \int_0^t \frac{x - \xi_2 \tau}{(\tau - r)^2} e^{-\frac{r^2}{4\mu(t-\tau)}} d\tau.$$

Here:

$$I_1 = \iint \bar{\omega}_1 d\xi d\eta, \quad I_2 = \iint \bar{\omega}_2 d\xi d\eta.$$

If the vortices move in circles about a common axis, and if the distance of the point xy from this axis is large compared with the distances of the vortices from the axis, we can put $\xi_1 = \eta_1 = \xi_2 = \eta_2 = 0$ and get:

$$u(x, y, t) = - \frac{I_1 + I_2}{2\pi} \frac{y}{R^2} \left(1 - e^{-\frac{R^2}{4\mu t}} \right),$$

$$v(x, y, t) = \frac{I_1 + I_2}{2\pi} \frac{x}{R^2} \left(1 - e^{-\frac{R^2}{4\mu t}} \right),$$

with R the distance of point xy from the axis. In distant points the motion behaves as if we had a vortex with intensity $I_1 + I_2$ at $t = 0$.